REAL FORMS OF SIMPLE LIE ALGEBRAS

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CERTIFICATE

This is to certify that the project work embodied in the dissertation **"Real forms of simple Lie algebras"** which is being submitted by **Kaushalya Rani Hota, Roll No.412MA 2065**, has been carried out under my supervision.

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ABSTRACT

This thesis is about the real forms of simple Lie algebras. Firstly we start with some basic theory of Lie algebra and different types of Lie algebras with some examples in Chapter 2. Thereafter in Chapters 3 to 4 we give the complete classification of real forms of simple Lie algebras. Also we have discuss the Chevalley basis which is important for the Simple Lie algebras. This project introduces Lie groups and their associated Lie algebras. In this thesis we introduce various properties of real forms, conjugations, and automorphisms of complex simple Lie algebras. Finally, there is a complete classification of real forms of simple Lie algebras.

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CHAPTER-1

1 INTRODUCTION

Lie algebras, and Lie groups, are named after Sophus Lie (pronounced lee), a Norwegian mathematician who lived in the latter half of the 19th century. He studied continuous symmetries (i.e., the Lie groups above) of geometric objects called manifolds, and their derivatives (i.e., the elements of their Lie algebras). They were not only interesting on their own right but also played an important role in twentieth century mathematical physics.

Today, more than a century after Lie's discovery, we have a vast algebraic theory studying objects like Lie algebras, Lie groups, Root systems,Weyl groups, Linear algebraic groups, etc.The study of the general structure theory of Lie algebras, and especially the important class of simple Lie algebras, was essentially completed by Elie Car- tan and Wilhelm Killing in the early part of the 20th century. The concepts introduced by Cartan and Killing are extremely important and are still very much in use by mathematicians in their research today.

In this thesis we discuss the classification of simple Lie algebras. It depends on the characteristic of the field and the complete classification for arbitrary characteristic is yet unknown. While the characteristic zero case was completely resolved many years ago, there are still open questions about the classification in positive characteristic. More precisely, the characteristics 2 and 3 seem to be very difficult and not much is known besides some examples of simple Lie algebras. Despite the difficulties, however, the classification of simple Lie algebras over fields of characteristic strictly greater than 3 has been recently completed. The aim of this thesis is to introduce the reader to the classification of simple Lie algebras done so far. We give in full the classification in characteristic zero and outline the basics, required to state the classification theorems for a positive characteristic known so far.

We discuss various properties of real forms, conjugations, and automorphisms of complex simple Lie algebras g. Some of these are well known, some not so well known, and some are new. For instance we show that the group G of all automorphisms of g, considered as a real Lie algebra, is a semidirect product of the group G of all automorphisms of g, considered as a complex Lie algebra. We exhibit several analogies between the compact real forms on one hand and the split real forms on the other hand. We also study pairs (and triples) of commuting conjugations of g with some additional properties.

CHAPTER-2

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2 LIE ALGEBRAS

The definition of a Lie algebra consists essentially of two parts revealing its structure. A Lie algebra L is firstly a vector space (linear space), secondly there is defined on L a particular kind of binary operation, i.e., a mapping $L \times L \to L$, denoted by $[.,.]$. The dimension of a Lie algebra is by definition the dimension of its vector space. It may be finite or infinite. The vector space is taken either over the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We will use F to denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1 A Lie algebra L is a vector space with a binary operation.

$$(x, y) \in L \times L \longrightarrow [x, y] \in L$$

called Lie bracket or commutator, which satisfies

1. For all $x, y \in L$ one has

$$[x, y] = -[x, y] \qquad \text{(antisymmetry)}$$

2. The binary operation is linear in each of its entries:

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z] \qquad \text{(bilinearity)}$$

For all $x, y \in L$ and all $\alpha, \beta \in F$ 3. For all $x, y, z \in L$ one has

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi identity)}$$

A Lie algebra is called real or complex when its vector space is respectively real or complex $(F = \mathbb{C})$.

Definition 2.1.2 A Lie algebra L is called abelian or commutative if $[x, y] = 0$ for all x and y in L

Definition 2.1.3 A subset K of a Lie algebra L is called a subalgebra of L if for all $x, y \in K$ and all $\alpha, \beta \in F$ one has $\alpha x + \beta y \in K$, $\quad [x, y] \in K$.

Definition 2.1.4 An ideal I of a Lie algebra L is a subalgebra of L with the property.

$$[I, L] \subset I$$

i.e for all $x \in I$ and all $y \in L$ one has

$$[x, y] \in I$$

Note

Every (non-zero) Lie algebra has at least two ideals namely the Lie algebra L itself and the subalgebra 0 consisting of the zero element only.

$$0 \equiv \{0\}$$

Both these ideal are called trivial. All non-trivial ideal are called proper.

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2.1 General linear Lie algebra

Let $L(V)$ be the set of all linear operators on a vectors space V . This vector space V will often be a finite-dimensional one.
We have for all $\alpha, \beta \in F$ and all $\nu \in V$

$$(\alpha A + \beta B)\nu = \alpha(A\nu) + \beta(B\nu)$$

and

$$(AB)\nu = A(B\nu)$$

clearly , $L(V)$ is an associative algebra with unit element 1; defined by $1\nu = \nu$ for all ν in V.
Defining on $L(V)$ a bracket operation by

$$[A, B] := AB - BA$$

turns $L(V)$ into a Lie algebra. This Lie algebra is called general Lie algebra $gl(V)$.

2.2 Derived algebra of a Lie algebra

Consider in L the set $L^{'} = [L, L]$. this is the set of element of the form $[x, y](x, y \in L)$ and possible linear combinations of such elements. It is called the derived algebra of L.

Lemma. Let

$$L^{'} := [L, L]$$

then this is an ideal in L.

Proof. The derived algebra $L^{'}$ is by definition a subspace of L.

since $L^{'} \subset L$

We have $[L^{'}, L^{'}] \subset [L, L] = L^{'}$ in order to prove that $L^{'}$ is an ideal. However from above it follows

$$[L^{'}, L] \subset [L, L] = L^{'} \qquad .$$

Definition 2.3.1 Let L be a Lie algebra. The sequence $L^0, L^1, ..., L^n, ...$ defined by

$$L^0 := L, L^1 := [L, L], L^2 := [L, L^1], ...$$

$$L^n := [L, L^{n-1}], ...$$

is called the descending central sequence.

Definition 2.3.1 Let L be a Lie algebra. The sequence $L_0, L_1, ..., L_n, ...$ defined by

$$L_0 := L, L_1 := [L_0, L_0], ..., L_n := [L_{n-1}, L_{n-1}], ...$$

is called the derived sequence.

Definition 2.3.2 A Lie algebra L is called nilpotent if there exist an $n \in N$ such that $L^n = 0$.

Remark. If $L^n = 0$ then of course $L^q = 0$ for $q \geq 0$. Let $L \neq 0$ and let p be the smallest integer for which $L^p = 0$. Then We have $[L, L_{p-1}] = 0$ and $L^{p-1} \neq 0$. This means that L^{p-1} is an abelian ideal in L. Hence each nilpotent Lie algebra $L \neq 0$ contains an ideal unequal to 0.

Definition 2.3.3 A Lie algebra L is called solvable if $L_n = 0$ for some $n \in N$.

Definition 2.3.4 The maximal solvable ideal of a Lie algebra L is called the radical of L and it is denoted by $R \equiv radL$.

2.3 Simple and Semisimple Lie algebras

Definition 2.4.1 A Lie algebra L is called simple if L is non-abelian and has no proper ideals.

Corollary. For a simple Lie algebra L the derived algebra $L^{'} = [L, L]$ is equal to L.

Proof. The derived algebra $L^{'}$ is an ideal. Since L is simple, $L^{'}$ has to be a trivial ideal. One has only two alternatives,either $L^{'} = 0$ or $L^{'} = L$. The first option is rule out since L is non-abelian. Therefore $L^{'} = L$.

Example. Consider in $gl(n)$ the subset of matrices with trace equal to zero. This is a Lie algebra called the $sl(n)$ and which is simple.

Definition 2.4.2 Lie algebra L is called semisimple if $L \neq 0$ and L has no abelian $ideal \neq 0$.

Levi's theorem. Let L be a finite-dimensional Lie algebra and $R \equiv radL$ its radical. Then there exists a semisimple subalgebra S of L such that L is the direct sum of its linear subspaces R and S.

$$L = R \oplus S$$

2.4 Idealizer and Centralizer

Definition 2.5.1 Let k be a subalgebra of a Lie algebra L. The idealizer $N_L(k)$ of k in L is defined as

$$N_L(k) := \{x \in L \mid \forall y \in k : [x, y] \in K\} \qquad .$$

Definition 2.5.2 A subalgebra k is called self-idealizing in L if $N_L(k) = k$.

Definition 2.5.3 A subalgebra of L. The centralizer $C_L(k)$ of k is defined by

$$C_L(k) := \{x \in L \mid \forall y \in k : [x, y] = 0\} \qquad .$$

Definition 2.5.4 The centralizer $C_L(L)$ of L itself is called the center of L. It is usually denoted by $Z(L) \equiv C_L(L)$ and it is given by

$$Z(L) := \{x \in L \mid \forall y \in L : [x, y] = 0\}$$

2.5 Derivations of a Lie algebra

Definition 2.6.1 A derivation δ of a Lie algebra L is a linear map
$\delta : L \longrightarrow L$ satisfying

$$\delta[x, y] = [\delta x, y] + [x, \delta y]$$

The collection of all derivation of L is denoted by $DerL$

2.6 Structure constants of a Lie algebra

Let L be a finite dimensional Lie algebra . Let n be the dimension of L. $\{e_1, e_2, ..., e_n\}$ is a basis for L. the every element of Lie algebra can be written as

$$x = \Sigma_{i=1}^{n} x^i e_i$$

Let $y \in L$, $y = \Sigma_{k=1}^{n} y^k e_k$

$$[x, y] = [\Sigma_{i=1}^{n} x^i e_i, \Sigma_{k=1}^{n} y^k e_k] = \Sigma_{i,k=1}^{n} x^i y^k [e_i, e_k]$$

So commutator $[x, y]$ of two element $x, y \in L$ is completely determine by the lie bracket $[e_i, e_k]$ of pairs of basis element $[e_i, e_k] \in L$. hence $[e_i, e_k]$ can again expanded w.r.t. the basis $\{e_1, e_2, ..., e_n\}$.

2.7 Special linear Lie algebra

The structure of the special linear Lie algebra $sl(n, \mathbb{C})$.

Definition 2.8.1 $sl(n, \mathbb{C})$ is the set of all $n \times n$ matrics with complete entries having trace zero. The lie bracket of element of $sl(n, \mathbb{C})$ is commutator of there matrices.

$$dim(sl(n, \mathbb{C})) = n^2 - 1$$

Remark. We are interested for matrices $n \geq 2$ putting $n = k + 1$ for $k \geq 1$ with this convention the special linear lie algebra is denoted by $asl(k + 1, \mathbb{C})$ or A_k.

The dimension of A_k is

$$dim(sl(k + 1, \mathbb{C}))(k + 1)^2 - 1 = k^2 + 2k + 1 - 1 = k(k + 2)$$

2.8 Lie groups and Lie algebras

We discuss briefly the relationship between Lie algebras and Lie groups. Since we will be dealing with linear Lie groups, i.e. groups the elements of which are linear operators on some vector space.

Our discussion is based on the complex general linear Lie groups $GL(V)$, the groups of bijective linear operators on a complex $n-$ dimensional vector space V. Denoting the Groups elements by capitals A, B etc. We define a matrix representation of these operators by taking a basis in V. Then the matrix (a_{ij}) representing the operators A is defined by

$$Ae_i = \Sigma_{j=1}^{n} e_j a_{ji} \qquad (i = 1, 2, ..., n)$$

In this way we obtain the isomorphism $A \in GL(V) \longrightarrow (a_{ij}) \in GL(n, \mathbb{C})$ where $GL(n, \mathbb{C})$ is the groups of all invertible $n \times n$ matrices. Next we indicate why both groups, $GL(V)$ and

$GL(n,\mathbb{C})$, are n^2- dimensional complex Lie groups. Let $M(n,\mathbb{C})$ be the set of all complex $n\times n$ matrices, then map

$$k:(a_{ij})\in M(n,\mathbb{C})\longrightarrow(a_{11},a_{12},...,a_{nn})\in\mathbb{C}^{n^2}$$

is a bijection. From the fact that the map

$$det:(a_{ij})\longrightarrow det(a_{ij})\in\mathbb{C}$$

is a continuous function of the matrix elements it follows that $GL(n,\mathbb{C})=\{A\in M(n,\mathbb{C}\ |\ detA\neq 0\}$ is an open set in $\mathbb{C}^{n^2}$.

This implies that the restriction $k|GL(n,\mathbb{C})$ map the open set $GL(n,\mathbb{C})$ bijectively onto the open set $\mathbb{C}^{n^2}|k$, where

$$k:\{(a_{11},a_{12},...,a_{nn})\in\mathbb{C}^{n^2}|det(a_{ij})=0$$

In the general theory of Lie groups it is shown that the vector space structure of the Lie algebra of a Lie groups is isomorphic with the tangent space at the unit element of the group manifold. For a linear Lie group the tangent space is readily obtained. Consider in $GL(n,\mathbb{C})$ a subset of operators $A(t)$depending smoothly on areal parameter t and such that $A(0)=1$ where 1 is the identity operators on V. Such a subset is called a curve through the unit element. The tangent vector at $t=0$ is obtained by making the Taylor expansion of $A(t)$ upto the first order term.

$$A(t)=A(0)+M(t)+O(t^2)$$

with M the derivative of $A(t)$ at $t=0$;

$$M=A(0)$$

Next we define subgroups of $GL(n,\mathbb{C})$ by considering subset of elements in $GL(n,\mathbb{C})$ that leave a specific non-degenerate bilinear form on the vector space V invariant.

A bilinear form , denotes by $(.,.)$ is a map $(.,.):V\times V\longrightarrow\mathbb{C}$ such that for all $\alpha,\beta\in\mathbb{C}$ and all $x,y,z\in V$ one has

$$(\alpha x+\beta y,z)=\alpha(x,z)+\beta(y,z)$$

and

$$(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)$$

A bilinear form is called symmetric if for all $x, y \in V$

$$[x, y] = [y, x]$$

A bilinear form is called skew-symmetric if $\forall x, y \in V$

$$[x, y] = -[y, x]$$

A skew-symmetric form $(.,.)$ is called non-degenerate
if $(x, y) = 0 \ \forall y \in V \Rightarrow x = 0$
We will now point out that a bilinear form $(.,.)$ on a vector space V can be used to define a subgroup of $GL(n, \mathbb{C})$ consider in $GL(n, \mathbb{C})$ the subset S of elements which leave invariant the form $(.,.)$. An elements $A \in Gl(n, \mathbb{C})$ is said to leave form $(.,.)$ invariant if we have for all $x, y \in V$

$$(Ax, Ay) = (x, y)$$

Let $A, B \in S$ then

$$(ABx, ABy) = (Bx, By) = (x, y)$$

This means that the product $AB \in S$. From the fact that the unit operators 1 belongs to S one obtains as follows that $A \in S$ implies $A^{-1} \in S$.

$$(x, y) = (1x, 1y) = (AA^1x, AA^1y) = (A^{-1}x, A^{-1}y)$$

We conclude that the set S is a subgroup of $GL(n, \mathbb{C})$ The condition of invariance (1) on the groups can be translated to a condition on the Lie-algebra. Taking instead of A a curve $A(t) = expMt$, one has Differentiation of this express with respect to t and taking $t = 0$ gives

$$(Mx, y) + (x, My) = 0$$

Example. The simplest examples of an r parameter Lie groups is the abelian Lie group R^r . The groups operation is given by vector addition. The identity element is the zero vector and the inverse of a vector x is the vector $-x$.

CHAPTER-3

3 CLASSIFICATION OF SIMPLE LIE ALGEBRAS

3.1 Cartan matrix

Definition: The cartan matrix $(A_{ij})_{i,j=1}^{k}$ of semisimple Lie algebra is defned by means of the dual contraction between Π and Π^{v}.

$$A_{ij} = \langle \alpha_j, \alpha_i^v \rangle = \frac{2\,(\alpha_j \mid \alpha_i)}{(\alpha_i \mid \alpha_i)} \tag{1}$$

The matrix elements of the cartan matrix are the cartan integers of simple roots. An immediate consequence of (1) is a relation between the ratios of lengths of simple roots and matrix elements of the caran matrix

$$\frac{A_{ij}}{A_{ji}} = \frac{\|\alpha_j\|^2}{\|\alpha_i\|^2}$$

3.2 Cartan Subalgebra

Definition: A subalgebra $K \subset L$ is called a cartan subalgebra of L if K is nilpotent and equal to its own normalizer.

The adjoint representation

For every $x \in L$ a linear operator adx on the vector space L by means of the Lie bracket on L, namely $adx := [x, y]$, for all $y \in L$ the map
$(x, y) \in L \times L \longrightarrow adx(y) \in L$

$$ad[x, y] = [adx, ady]$$

$$ad[x, y](z) = [[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

$$= (adxady)(z) - (adyadx)(z)$$

$$= [adx, ady](z)$$

$$ad : x \in L \longrightarrow adx \in gl(L)$$

is a representations of L with representation space L. This representation is called the adjoint representations.

$$ade_i(e_j) = [e_i, e_j] = \sum_k c_{ij}^k e_k$$

This yields a matrix representation of ade_i.

3.3 Cartan-killing form

Definition. Let L be a Lie algebra.The map $K : L \times L \longrightarrow F$ given by $K(x, y) = Tr(adxady)$

Properties

1. $K(\alpha x + \beta y, z) = \alpha K(x, z) + \beta K(y, z)$

   $K(x, \alpha y + \beta z) = \alpha K(x, y) + \beta K(x, z)$ (bilinearity)

2. $K(x, y) = K(y, x)$ (symmetry)

3. $K([x, y], z) = K(x, [y, z])$ (associativity)

3.4 Root Space Decomposition

Let H be a maximal toral subalgebra of L. The linear Lie algebra $ad_L H$ is a commuting set of digonalizable linear operators on the vector space L. So L has a basis consisting of the simulteneous eigen vectors of the set operators $\{adh \mid h \in H\}$. This means that L decomposes into a direct sum of subspace.

This subspace which will be denoated by L_α. The vectors $x \neq 0$ in $L_\alpha \subset L$ aer by definition eigen vectors of adh for all $h \in H$. Denoting eigen value of adh by $\alpha(h)$ one has for all $h \in H$.

$adh(x) = \alpha h(x)$

Clearly the lable α is the function $\alpha : h \in H \to \alpha(h) \in \mathbb{C}$ which associates the eigenvalues $\alpha(h)$ of the eigen vector x to the element h.

Definition. Let H be a maximal toral sub algebra of a finite dimensional complex semisimple Lie algebra L. The eigenvalues of the linear operator adh will be denoated by αh and one defines the subspace L_α of L by

$L_\alpha := \{x \in L \mid \forall \ h \in H : adh(x) = \alpha(h)x\}$

Then the Lie algebra L is a vector space direct sum of the subspace L_α :

$L = \bigoplus_\alpha L_\alpha$

This is called the rootspace decomposition of L w.r.t H.

3.5 Different types of Simple Lie algebra (A_n, B_n, C_n, D_n)

Without loss of generality we shall work over C in this entire subsection. We consider the classical Lie algebras $sl(n, \mathbb{C}); so(n, \mathbb{C})$ and $sp(n, \mathbb{C})$ for $n \geq 2$. We want to find their root systems and to show that their Dynkin diagrams.

$A_n - Type(Sl(n+1, \mathbb{C})$

(1) The root space decomposition of $L = Sl(n+1, \mathbb{C})$ is

$$L = H \oplus \bigoplus_{i \neq j} L_{\varepsilon_i - \varepsilon_j}$$

Where $\varepsilon_i(h)$ is the $i-$th entry of h and the root space $L_{\varepsilon_i - \varepsilon_j}$ is spanned by e_{ij}. Thus $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq n+1\}$.

(2) If $i < j$, then we have $[e_{ij}, e_{ji}] = e_{ii} - e_{jj} = h_{ij}$ and $[h_{ij}, e_{ij}] = 2e_{ij} \neq 0$ and thus $[[L_\alpha, L_{-\alpha}], L_\alpha]$ for each root $\alpha \in \phi$.

(3) The root system Φ has as a base $\{\alpha_i : 1 \leq i \leq n\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

(4) We have already computed the Cartan matrix for this root system. We have simply the following

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i = j \\ -1, & if \, |i-j| = 1; \\ 0, & otherwise \end{cases}$$

We shall also notice that from (2) follows that the standard basis elements for the subalgebra $Sl(\alpha_i)$ can be taken as

$$e_{\alpha_i} = e_{i,i+1}, \;\; f_{\alpha_i} = e_{i+1,i}, \;\; h_{\alpha_i} = e_{i,i} - e_{i+1,i+1}$$

Calculation shows that .Here L is simple. We say that the root system of $sl(n+1, \mathbb{C})$ has type A_n.
and the Dynkin diagram is :

$$\begin{array}{ccccccc} \bigcirc & - - - - & \bigcirc & - - - - \cdots - & \bigcirc & - - - - & \bigcirc \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array}$$

. This diagram is connected, so L is simple.

B_n**-Type** $(So(2n+1, \mathbb{C}))$

$so(2n+1, \mathbb{C})$ is represented by the block matrices of the type

$$x = \begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix},$$

with $p = -p^t$ and $q = -q^t$. As usual, let H be the set of diagonal matrices in L. We label the matrix entries from 0 to $2n$ and thus every element $h \in H$ can be written in the form $h = \Sigma_{i=1}^n a_i(e_{ii} - e_{i+n,i+n})$, where $0, a_1, ...a_n, -a_1, ... - a_n$ are exactly the diagonal entries of h.

(1) We first start by finding the root spaces for H and then using them we find the root space decomposition of L. Now consider the subspace of L spanned by the matrices whose non-zero entries lie only in the positions labeled by b and c. Now using our labeling and looking at the block matrix above we easily see that this subspace has a basis $b_i = e_{i,0} - e_{0,n+i}$ and $c_i = e_{0,i} - e_{n+i,0}$ for $1 \leq i \leq n$.
We do the following calculation:

$$[h, b_i] = [\Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n}), e_{i,0} - e_{0,n+i}]$$

$$= \Sigma_{i=1}^{n} a_i([e_{ii}, e_{i,0}] - [e_{ii}, e_{0,n+i}] + [e_{n+i,n+i}, e_{i,0}])$$

$$= \Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n}) = a_i b_i$$

where we use the following relations
$[e_{ii}, e_{i,0}] = e_{i0}$, $[e_{ii}, e_{0,n+i}] = 0$, $[e_{n+i,n+i}, e_{i,0}] = 0$, $[e_{n+i,n+i}, e_{0,n+i}] = -e_{n+i,n+i}$.
Similarly, we get $[h, c_i] = -a_i c_i$. Further, we extend to a basis of L by the matrices:

$$m_{ij} = e_{ij} - e_{n+j,n+i} \qquad \text{for } 1 \leq i \neq j \leq n,$$

$$p_{ij} = e_{i,n+j} - e_{j,n+i} \qquad \text{for } 1 \leq i < j \leq n,$$

$$q_{ij} = p_{ij}^t = e_{n+j,i} - e_{n+i,j} \qquad \text{for } 1 \leq i < j \leq n.$$

We now calculate the following relations:

$$[h, m_{ij}] = (a_i - a_j)m_{ij},$$

$$[h, p_{ij}] = (a_i + a_j)p_{ij},$$

$$[h, q_{ij}] = -(a_i + a_j)q_{ji}.$$

We can now list the roots. For1 $\leq i \leq n$, let $\varepsilon_i \in H^*$ be the map sending h to a_i, its entry position i.

(2) It suffices to show that $[h_\alpha, x_\alpha] \neq 0$, where $h_\alpha = [x_\alpha, x_{-\alpha}]$. We do this in three steps.

First, for$\alpha = \varepsilon_i$, we have $h_i = [b_i, c_i] = e_{ii} - e_{n+i,n+i}$ and by (1) we have $[h_i, b_i] = b_i$. Second, for $\alpha = \varepsilon_i - \varepsilon_j$ and$i < j$, we have$h_{ij} = [m_{ij}, m_{ji}] = (e_{ii} - e_{n+i,n+i}) - (e_{jj} - e_{n+j,n+j})$ and again by (1) we obtain $[h_{ij}, m_{ij}] = 2m_{ij}$. Finally, for $\alpha = \varepsilon_i + \varepsilon_j$ and $i < j$, we have $k_{ij} = [p_{ij}, q_{ji}] = (e_{ii} - e_{n+i,n+i}) + (e_{jj} - e_{n+j,n+j})$ whence $[k_{ij}, p_{ij}] = 2p_{ij}$.

(3) The base for our root system is given by $\Delta = \}\alpha_i : 1 \leq i < n\} \cup \}\beta_n$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\beta_n = \varepsilon_n$. For $1 \leq i < n$ we see that $\varepsilon_i = \alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \beta_n$
and for $1 \leq i < j \leq n$,
$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$,
$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} + 2(\alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \beta_n)$.
Now using our table of roots we see that if $\gamma \in \Phi$ then either γ or γ appears above as a non-negative linear combination of elements of Δ. Since $dimH$ is the same as the number of elements of Δ, precisely n ,we conclude that Δ is a base for Φ.

(4) For $i < j$ take $e_{\alpha_i} = m_{i,i+1}$ and by (2) follows $h_{\alpha_i} = h_{i,i+1}$. Taking $e_{\beta_n} = \beta_n$ we see that $h_{\beta_n} = 2(e_{nn} - e_{2n,2n})$. For $1 \leq i, j \leq n$, we calculate that

$$[h_{\alpha_j}, e_{\alpha_i}] = \begin{cases} 2e_{\alpha_i}, & i = j \\ -e_{\alpha_i}, & if\, |i-j| = 1; \\ 0, & otherwise \end{cases}$$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2, & i = j \\ -1, & if\, |i-j| = 1; \\ 0, & otherwise \end{cases}$$

Similarly, by calculating $[h_{\beta_n}; e_{\alpha_i}]$ and $[h_{\alpha_i}; e_{\beta_n}]$, we find that

$$\langle \alpha_i, \beta_n \rangle = \begin{cases} -2, & i = n-1 \\ 0, & otherwise \end{cases}$$

$$\langle \beta_n, \alpha_i \rangle = \begin{cases} -1, & i = n-1 \\ 0, & otherwise \end{cases}$$

.

This shows that the Dynkin diagram of ϕ is :

$$\underset{\alpha_1}{\bigcirc} - \underset{\alpha_2}{\bigcirc} - - - - - - - - \cdots - - - - - - \underset{\alpha_{n-1}}{\bigcirc} \Longrightarrow \underset{\alpha_n}{\bigcirc} \qquad (2)$$

and since it is connected, ϕ is irreducible and so L is simple. The root system of $so(2n+1,\mathbb{C})$ is said to have type B_n.

C_n**-Type** $(sp(2n,\mathbb{C}))$

The elements of this algebra as block matrices as follows:

$$\begin{pmatrix} m & p \\ q & m^t \end{pmatrix}$$

where $p = p_t$ and $q = q_t$. The first observation to make is that for $n = 1$ we have $sp(2,\mathbb{C}) \cong sl(2,\mathbb{C})$, since we have 2×2 matrices with entries numbers and not block matrices. Thus, without loss of generality we will assume that $n > 2$. As above, H is the set of diagonal matrices in L. We also use the same labeling of the matrix entries so $h = \Sigma_{i=1}^{n} a_i(e_{ii} - e_{i+n,i+n})$.

(1) Take the following basis for the root space of L:

$$m_{ij} = e_{ij} - e_{n+j,n+j} \qquad \text{for } 1 \leq i \neq j \leq n,$$

$$p_{ij} = e_{i,n+j} - e_{j,n+j} \qquad \text{for } 1 \leq i < j \leq n, \quad p_{ii} = e_{i,n+i} \quad \text{for } 1 \leq i \leq n$$

$$q_{ij} = p_{ij}^t = e_{n+j,i} - e_{n+i,j} \qquad for 1 \leq i < j \leq n \qquad .$$

Calculations show that:

$$[h; m_{ij}] = (a_i - a_j)m_{ij},$$

$$[h; p_{ij}] = (a_i + a_j)p_{ij},$$

$$[h; q_{ij}] = -(a_i + a_j)q_{ji}.$$

Clearly, for $i = j$ the eigenvalues for p_{ij} and q_{ji} are $2a_i$ and $-2a_i$ respectively.

(2) Now we must check that $[h, x_\alpha] \neq 0$ with $h = [x_\alpha; x_{-\alpha}]$ holds for each root . It has been done for $\alpha = \varepsilon_i - \varepsilon_j$ for $so(2n+1, \mathbb{C})$. If $\alpha = \varepsilon_i + \varepsilon_j$, then $x_\alpha = p_{ij}$ and $x_{-\alpha} = q_{ji}$ and $h = (\varepsilon_{ii} - \varepsilon_{l+i,l+i}) + (\varepsilon_{jj} - e_{n+j,n+j})$ for $i = j$. We then have $[h, x_\alpha] = 2x_\alpha$ in both cases.

(3) Choose $\alpha_i = \varepsilon_i - \varepsilon_{i+1} for 1 \leq i \leq n-1$ as before, and $\beta_n = 2\varepsilon_n$. Our claim now is that $\{\alpha_1, ..., \alpha_{n-1}, \alpha_n\}$ is a base for the root system ϕ of $sp(2n, \mathbb{C})$. For $1 \leq i < j \leq n$ we have $\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$, $\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + ... + \alpha_{n-1} + \beta_n)$, $2\varepsilon_i = \alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \beta_n$,
Thus using the same arguments as above we conclude that $(\alpha_1, ... + \alpha_{n-1}, \beta_n)$ is the base of ϕ.

(4) In the end we need to calculate the Cartan integers. The numbers $\langle \alpha_i, \alpha_j \rangle$ are al- ready known. Taking $(e_{\beta_n} = p_{nn}$ we find that $h_{\beta_n} = e_{n,n} - e_{2n,2n}$ and so.

$$\langle \alpha_i, \beta_n \rangle = \begin{cases} -1, & i = n-1 \\ 0, & otherwise \end{cases}$$

$$\langle \beta_n, \alpha_j \rangle = \begin{cases} -2, & i = n-1 \\ 0, & otherwise \end{cases}$$

The Dynkin diagram of this root system.

$$C_n \qquad \underset{\alpha_1}{\bigcirc} - - - - - \underset{\alpha_2}{\bigcirc} - - - - - - \cdots - \underset{\alpha_{n-1}}{\bigcirc} \Longleftarrow \underset{\beta_n}{\bigcirc}$$

which is connected, so L is simple. The root systems of $sp(2n, \mathbb{C})$ is said to have type C_n.

D_n**-Type** $(so(2n, \mathbb{C}))$

All the elements of this classical algebra as block matrices:
where $p = -p^t$ and $q = -q^t$.
We observe that for$n = 1$ our Lie algebra is one dimensional so by definition is neither simple nor semisimple. In particular, the classical Lie algebra $so(2, \mathbb{C})$ is neither simple or semisimple. Again H is the set of diagonal matrices in L and we do the same labeling as in the former case. Thus we can use the calculations above by simply ignoring the row and column of matrices labeled by 0.

(1) We now simply copy the second half of the calculations for $so(2n + 1, \mathbb{C})$.

(2) The calculations done above immediately yield that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root α.

(3) We now claim that the base for our root system is $\Delta = \{\alpha_i : 1 \leq i < n\} \cup \{\beta_n\}$, where $\alpha = \varepsilon_i - \varepsilon_{i+1}$ and $\beta = \varepsilon_{n-1} - \varepsilon_n$. For $1 \leq i < j \leq n$, we have the following:
$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$,
$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + ... + \alpha_{n-2} + (\alpha_j + \alpha_{j+1} + ... + \alpha_{n-1} + \beta_n)$.
Then if $\gamma \in \phi$ then either γ or $-\gamma$ is a non-negative Z-linear combination of elements of Δ. Therefore, Δ is a base for our root system.

(4) Now we calculate the Cartan integers. The work already done for $so(2n+1, \mathbb{C})$ gives us the Cartan numbers $\langle \alpha_i, \alpha_j \rangle$ for $i, j < n$. To calculate the remaining ones we take $e_{\beta_n} = p_{n-1,n}$ and use (2) from $so(2n+1; C)$. Thus we obtain that $h_{\beta_n} = (e_{n-1,n-1} - e_{2n-1,2n-1}) + (e_{n,n} - e_{2n,2n})$. Hence

$$\langle \alpha_j, \beta_n \rangle = \begin{cases} -1, & j = n-2 \\ 0, & otherwise \end{cases}$$

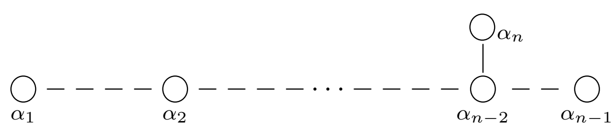
$$\langle \beta_n, \alpha_j \rangle = \begin{cases} -2, & j = n-2 \\ 0, & otherwise \end{cases}$$

.

If $n = 2$, then the base has only two orthogonal roots α_1 and β_2, so in this case , ϕ is reducible and hence $so(4, \mathbb{C})$ is not simple. If $n > 3$.

If $n = 2$, then the base has only two orthogonal roots α_1 and β_2, so in this case , ϕ is reducible and hence $so(4, \mathbb{C})$ is not simple. If $n > 3$, then our calculations show that the Dynkin diagram of ϕ is .

D_n



(4)

As this diagram is connected, the Lie algebra is simple. When $n \geq 3$, the Dynkin diagram is the same as A_3, the root system of $sl(4, \mathbb{C})$, so we have that $so(6, \mathbb{C}) \cong sl(4, \mathbb{C})$.For $n > 4$, the root system of $so(2n, \mathbb{C})$ is said to have type D_n. So far we have that only $so(2, \mathbb{C})$ and $so(4, \mathbb{C})$ are not simple. Therefore, it now remains to show that $sp(2n, \mathbb{C})$ is simple .

CHAPTER-4

4 REAL FORMS OF SIMPLE LIE ALGEBRAS

4.1 Real form

Let g be a Lie algebra. A real form of g is a Lie algebra $g_{\mathbb{R}}$ over $\mathbb{R}$ such that there exists an isomorphism from g to $g_{\mathbb{R}} \otimes \mathbb{C}$. If we replace $\mathbb{C}$ with $\mathbb{R}$ in the definition of g, we obtain a real form $g_{\mathbb{R}}$ which is called split.A real form of g corresponds to a semi-linear involution of g. Let g be a complex Lie algebra. A mapping $T : g \longrightarrow g$ satisfying $T([x, y]) = [T(x), T(y)], T(x+y) = T(x)+T(y), T(\alpha x) = \alpha^* T(x)$ and $T^2 = Id$ for all $x, y \in g$ and for $\alpha \in \mathbb{C}$ is called a conjugation of g. If g is complex Lie algebra and T be a conjugation (semi-linear involution) of g, then $g_{\mathbb{R}} = \{x \in g \mid T(x) = x\}$ is a real form of g.

4.2 Some Results

• Every complex semi-simple Lie algebra has a compact (unique) real form (U), where $U = \sum_{\alpha\in\Delta} R(ih_\alpha) + \sum_{\alpha\in\Delta} R(e_\alpha - f_\alpha) + \sum_{\alpha\in\Delta} R(i(e_\alpha + f_\alpha))$

• Let g be an complex Kac-Moody algebra, C be a real form of it which is compact type.The conjugacy classes of real forms of non compact type of g are in bijection with the conjugacy classes of involutions on C. The correspondence is given by $C = K \oplus P \to K \oplus iP$ where K and P are the $\pm 1-$ eigen space of the involution. Further more every real form is either of compact type or of non-compact type.

• Let $g_{\mathbb{R}}$ be a real form of non-compact type. Let $g_{\mathbb{R}} = K \oplus P$ be Cartan decomposition. The Cartan Killng form is negative definite on K and positive definite on P.

4.3 Real form of $A_1 = sl(2, \mathbb{C})$

Consider the Chevelley basis $[h, e] = 2e, [h, f] = -2f, [e, f] = h$ For this algebra Chevelley generators are:

$$\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

The Cartan involution which generates the compact real form of A_1 is given by:

$$e \to -f, \;\; f \to -e, \;\; h \to -h$$

The compact real form of A_1 is generated by $\{e - f, i(e+f), ih\}$. Explicitly we can write $C = a_1(ih) + a_2(e-f) + ia_3(e+f)$, where $a_1, a_2, a_3 \in R$.

$$C = a_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + ia_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & -ia_1 \end{pmatrix}$$

This is a skew-hermitan matrix with trace zero. So the compact from of $A_1 = su(2)$.

Case-I: Let σ be an involutive automorphism on $su(2)$, i.e. $\sigma : su(2) \longrightarrow su(2)$ such that $\sigma(X) = \bar{X}$, where $X \in su(2)$. Under this automorphism

$$\begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & -ia_1 \end{pmatrix} \to \begin{pmatrix} -ia_1 & a_2 - ia_3 \\ -a_2 - ia_3 & ia_1 \end{pmatrix}$$

Compairing with the Cartan decomposition $K \oplus P$ we have

$$K = \begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix} \in so(2), \quad P = \begin{pmatrix} ia_1 & ia_3 \\ ia_3 & -ia_1 \end{pmatrix}$$

Thus by Weyl unitary trick

$$K + iP = \begin{pmatrix} -a_1 & a_2 - a_3 \\ -a_2 - a_3 & a_1 \end{pmatrix}$$

which is identified as $sl(2, \mathbb{R})$ and it is a non-compact real form of A_1.

Case-II: Now defining another automorphism

$$\sigma(X) = I_{1,1} X I_{1,1}$$

where the matrix $I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. we have

$$\sigma(X) = \begin{pmatrix} ia_1 & -a_2 - ia_3 \\ a_2 - ia_3 & -ia_1 \end{pmatrix}$$

Compairing with the Cartan decompositon $K \oplus P$ we have

$$K = \begin{pmatrix} ia_1 & 0 \\ 0 & -ia_i \end{pmatrix} \in so(2), \qquad P = \begin{pmatrix} 0 & a_2 + ia_3 \\ -a_2 + ia_3 & 0 \end{pmatrix}$$

Similarly now

$$K + iP = \begin{pmatrix} ia_1 & ia_2 - a_3 \\ -ia_2 - a_3 & -ia_1 \end{pmatrix} \in Su(1,1)$$

4.4 Real form of A_2

The Chevalley generators of A_2 are

$$\left\{ e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

Here the compact real form is generated by

$$\{(e_1 - f_1), (e_2 - f_2), (e_3 - f_3), i(e_1 + f_1), i(e_2 + f_2), i(e_3 + f_3), ih_1, ih_2\}$$

where

$$e_3 = [e_1, e_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, f_3 = [f_1, f_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The compact form is given by

$$\begin{pmatrix} ia_7 & a_1 + ia_4 & a_3 + ia_6 \\ -a_1 + ia_4 & -ia_7 + ia_8 & a_2 + ia_5 \\ -a_3 + ia_6 & -a_2 + ia_5 & -ia_8 \end{pmatrix},$$

The trace of the above matrix is zero satisfies the condition $A + A^* = 0$ which is identified as $su(3)$.

Case-I: Let $\sigma : su(3) \longrightarrow su(3)$ such that $\sigma(X) = \bar{X}$, where $X \in su(3)$. Under this automorphism

$$\begin{pmatrix} ia_7 & a_1 + ia_4 & a_3 + ia_6 \\ -a_1 + ia_4 & -ia_7 + ia_8 & a_2 + ia_5 \\ -a_3 + ia_6 & -a_2 + ia_5 & -ia_8 \end{pmatrix} \to \begin{pmatrix} -ia_7 & a_1 - ia_4 & a_3 - ia_6 \\ -a_1 - ia_4 & ia_7 - ia_8 & a_2 - ia_5 \\ -a_3 - ia_6 & -a_2 - ia_5 & ia_8 \end{pmatrix},$$

Compairing with the Cartan decomposition $K \oplus P$ we have

$$K = \begin{pmatrix} 0 & a_1 & a_3 \\ -a_1 & 0 & a_2 \\ -a_3 & -a_2 & 0 \end{pmatrix} \in so(3), P = \begin{pmatrix} ia_7 & ia_4 & ia_6 \\ ia_4 & -ia_7 + ia_8 & ia_5 \\ ia_6 & ia_5 & -ia_8 \end{pmatrix}$$

Thus by Weyl unitary trick

$$K + iP = \begin{pmatrix} -a_7 & a_1 - a_4 & a_3 - a_6 \\ -a_1 - a_4 & a_7 - a_8 & a_2 - a_5 \\ -a_3 - a_6 & -a_2 - a_5 & a_8 \end{pmatrix} \quad \in sl(3, \mathbb{R}).$$

Case-II: Now considering another automorphism

$$\sigma(X) = I_{2,1} X I_{2,1}$$

where the matrix

$$I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The compact form of A_2

$$\begin{pmatrix} ia_7 & a_1 + ia_4 & a_3 + ia_6 \\ -a_1 + ia_4 & -ia_7 + ia_8 & a_2 + ia_5 \\ -a_3 + ia_6 & -a_2 + ia_5 & -ia_8 \end{pmatrix},$$

can be written in this form

$$I_{2,1} = \begin{pmatrix} (A)_{2\times 2} & (B)_{2\times 1} \\ (-B^*)_{2\times 1} & (C)_{2\times 1} \end{pmatrix}.$$

Now applying the automorphism on compact form we have

$$I_{2,1} = \begin{pmatrix} A & B \\ (-B^*) & C \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ -B^* & C \end{pmatrix}$$

Here

$$K = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} ia_7 & a_1 + ia_4 & 0 \\ -a_1 + ia_4 & -ia_7 + ia_8 & 0 \\ 0 & 0 & -ia_8 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}.$$

K is isomorphic to $su(2) \times C_0 \times su(1)$ where C_0 is the centre of K. Thus

$$K + iP = \begin{pmatrix} ia_7 & a_1 + ia_4 & a_3 + ia_6 \\ -a_1 + ia_4 & -ia_7 + ia_8 & a_2 + ia_5 \\ -a_3 + ia_6 & -a_2 + ia_5 & -ia_8 \end{pmatrix} \quad \in su(2,1).$$

Similarly real form of A_n can be calculated which are listed below:

$A_n(sl(n+1,\mathbb{C}))$:

Compact real form: $su(n)$.

$su(p,q), \quad p+q=n+1, \quad p \geq q > 0, \quad p+q \geq 2.$

Non-Compact real form: $sl(n,\mathbb{R})$.

For n is even i.e., compact real form $su(2n)$: $so^*(2n), \quad n \geq 4.$

Where

$su(n) = \{A \in gl(n,\mathbb{C}) \mid A^* + A = 0, \quad \text{Trace } A = 1\}.$

$sl(n,\mathbb{R}) = \{A \in gl(n,\mathbb{R}) \mid \text{Trace } A = 0\}.$

$\text{Su}(p,q) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ {Z_2}^* & Z_3 \end{pmatrix} \middle| Z_1, Z_3 \text{ Skew Hermitan of order p and q respectively,} \right.$

$\left. Tr\ Z_1 + Tr\ Z_3 = 0, Z_2 \text{ is arbitrary} \right\}.$

$\text{So}^*(2n) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \middle| Z_1, Z_2 \ \ n \times n \text{ complex matrices,} \right.$

$\left. Z_1 \text{ skew}, Z_2 \text{ Hermitan} \right\}.$

Similarly real form of B_n can be calculated which are listed below:

$B_n(So(2n+1))$:

Compact real form : $So(2n+1)$.

Non-compact real form : $So(p,q), \quad p+q = 2n+1.$

with $p > q > 0, \quad p+q \geq 3 \quad \text{where} \quad p+q \quad \text{odd}.$

Similarly real form of C_n can be calculated which are listed below:

$C_n(Sp(n))$:

Compact real form : $Sp(n)$.

Non-compact real form :

$(i) \ \ Sp(n,\mathbb{R}), \quad n \geq 3.$

$(ii) \ \ Sp(p,q), \quad p+q=n, \quad p \geq q > 0, \quad p+q \geq 8, \quad p+q \ \text{ even}.$

Similarly real form of D_n can be calculated which are listed below:

$D_n(So(2n))$:

Compact real form : $So(2n)$.

Non-compact real form :

(i) $So(p,q), \quad p+q=2n.$

(ii) $So^*(2n)$ with $n \geq 4$.

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